

Elusive Order Parameters for Non-Abelian Gauge Theories

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(February 7, 2008)

Abstract

We construct a set of order parameters for non-Abelian gauge theories which probe directly the unbroken group and are free of the deficiencies caused by quantum fluctuations and gauge fixing which have plagued all previous attempts. These operators can be used to map out the phase diagram of a non-Abelian gauge theory.

PACS numbers:11.15.-q,03.65.Bz,03.80.+r

A convenient way to investigate the phase diagram of a gauge theory is to construct a set of order parameters which exhibit non-analytical behavior at phase boundaries. For *Abelian* gauge theories, Preskill and Krauss [1] have made a successful construction by invoking the Aharonov-Bohm effect [2] between cosmic strings and charged particles.¹ Generalization to non-Abelian gauge theories turned out to be very elusive. In spite of much progress in our understanding of the subtler aspects of non-Abelian gauge theories [4,6,7], it proves difficult to formulate a general procedure that unambiguously specifies the unbroken group. The key difficulty is quantum fluctuations. We need a framework that takes full account of the effects of virtual cosmic string loops (magnetic flux tubes). Any useful Aharonov-Bohm experiment necessarily proceeds in two stages: calibration and measurement. World sheets of virtual string loops can wrap around test charges, thus changing their states relative to other charges in the universe. Consequently, repeated flux measurements with test charges will not necessarily agree. It is also important for us to gauge fix. An element of a unbroken group has no invariant meaning unless the value of the Higgs field is specified. We address these important issues in this Letter and construct a set of order parameters that will probe directly and unambiguously the unbroken group of a non-Abelian gauge theory. We borrow our ideas from Alford *et al.* [4] who developed but immediately rejected a set of operators which were plagued by quantum fluctuations.

We emphasize that the idea of using the Aharonov-Bohm effect to probe the unbroken group is rather general. Moreover, the associated subtleties due to gauge fixing and quantum fluctuations of virtual magnetic flux tubes addressed in this Letter are general consequences of a partial symmetry breakdown of a non-Abelian gauge theory and the non-Abelian Aharonov-Bohm effect. These issues arise no matter the subgroup is discrete or continuous. However, for simplicity, we will discuss the following scenario only. Consider a

¹Another promising order parameter—the vacuum overlap order parameter—has been proposed by Fredenhagen and Marcu [3]. See [4,5] for discussions.

simply-connected gauge group K which is spontaneously broken into a discrete non-Abelian group G . This symmetry breaking leads to the existence of stable vortices (in 2+1 dimensions) and cosmic strings (in 3+1 dimensions) labelled by the elements of G . (Topologically stable vortices are classified by homotopy group $\pi_1(K/G)$ [8,9]. It follows from the exact homotopy sequence that $\pi_1(K/G) \simeq G$ for a simply connected K . To be more precise, the spectrum of stable vortices only spans G . An element of G may be unstable to decay into two or more vortices with the same total flux. This is, however, of no interest to our discussion.) Depending on its Higgs structure and the parameters of the Higgs potential, the symmetry group G may be further broken into a subgroup H . We would like to construct a set of order parameters to test if such a symmetry breakdown has occurred and if it does, what is the unbroken group H ?

In a free G charge phase, charged particles generally scatter off cosmic strings non-trivially. When a charged particle in the initial state $|u\rangle$ (and representation ν) winds around a string loop of flux a , its state becomes $D^\nu(a)|u\rangle$. This non-Abelian Aharonov-Bohm effect can be invoked to probe the phase diagram of a non-Abelian gauge theory. More concretely, we proceed as follows: prepare (or calibrate) a set of classical vortices, one for each element of G . Measure the non-Abelian Aharonov-Bohm phases acquired by charged particles which wind around the various vortices that we have prepared. Read off the spectrum of stable vortices from the results of our experiments and decide if symmetry breaking has occurred.

In the free charge phase, each element of G is associated with a stable pointlike vortex. Indeed, a test charge in the representation ν which traverses a g -string acquires a phase $D^\nu(g)$ as expected. We, therefore, conclude that the gauge group G is unbroken. On the other hand, if G is broken into a subgroup H , the elements of G that are not in H are not associated with isolated cosmic strings, but with strings that are boundaries of domain walls. Such domain walls are unstable and will decay via spontaneous nucleation of string loops [4,8,9]. Consider the insertion of a classical string worldsheet of flux $a \notin H$. Holes eventually appear in the wall bounded to the a string and collide with one another. Ultimately, the one

b with the least string tension will dominate the decay and a charged particle scattering off the composite string will therefore measure a flux $ab^{-1} \in H$ rather than a . Consequently, if we measure the Aharonov-Bohm phases acquired by test particles which traverse the various calibrated vortices, we find that the fluxes of the vortices are always elements of H and conclude that a symmetry breaking from G to H has occurred.

It was suggested in [10] that when a $U(1)$ gauge symmetry is spontaneously broken into Z_N , the discrete Z_N charge Q_Σ^* contained in a closed surface Σ^* can still be measured via the Gauss law:

$$F(\Sigma^*) = \exp\left(\frac{2\pi i}{N} Q_{\Sigma^*}\right) = \exp\left(\frac{2\pi i}{Ne} \int_{\Sigma^*} E \cdot ds\right). \quad (1)$$

($F(\Sigma^*)$ is closely related to the 't Hooft loop operator [11].) $F(\Sigma^*)$ inserts a classical cosmic string source on the world sheet Σ^* .

Now we turn to the operator which introduces classical charges into the system. An obvious choice would be the Wilson loop operator $W^\nu(C)$ where ν is an irreducible representation of the gauge group, G . The Aharonov-Bohm phase acquired by a charged particle which scatters off a cosmic string is represented by $F(\Sigma^*)W^\nu(C)$ where C and Σ^* have a non-trivial linking number. One might naively expect that $F(\Sigma^*)W^\nu(C)$ is the order parameter. This is not quite correct because quantum mechanical fluctuations near the surface Σ^* cause an area law decay of the *modulus* of $F(\Sigma^*) \sim \exp(-\kappa A(\Sigma^*))$. Fortunately, the *phase* of $F(\Sigma^*)$ remains unscreened and we can isolate it by dividing out its vacuum expectation value and obtain $\frac{F(\Sigma^*)}{\langle F(\Sigma^*) \rangle}$ [1]. Similarly, quantum fluctuations also lead to the exponential decay of the expectation value of $W(C)$. Therefore, the true order parameter for Abelian gauge theories is [1]

$$A^\nu(\Sigma^*, C) = \frac{F(\Sigma^*)W^\nu(C)}{\langle F(\Sigma^*) \rangle \langle W^\nu(C) \rangle}. \quad (2)$$

In the free Z_N charge phase, the order parameter (for the fundamental representation) gives

$$\lim \langle A(\Sigma^*, C) \rangle = \exp\left(\frac{2\pi i}{N} k(\Sigma^*, C)\right). \quad (3)$$

Here the limit is taken with Σ^* and C increasing to infinite size, and with the closest approach of Σ^* to C also approaching infinity; $k(\Sigma^*, C)$ denotes the linking number of the surface Σ^* and the loop C . (Note that other than these requirements, the value of A is independent of the details of Σ^* and C . Thus, we can probe the unbroken group by performing a finite number of thought experiments.) On the other hand, if there are no free Z_N charges, then we have

$$\lim \langle A(\Sigma^*, C) \rangle = 1. \quad (4)$$

The non-analytical behavior of $A(\Sigma^*, C)$ guarantees that the two phases are separated by a well-defined phase boundary. To probe the realisation of any Abelian discrete gauge symmetry, we just consider the operators $F_a(\Sigma^*)$ for each element $a \in G$.

When the gauge group is non-Abelian, the flux, h of a string has no gauge invariant meaning. One can imagine choosing an arbitrary base point x_0 , setting up a basis of test particles and calibrating the flux of a string by scattering these particles of known transformation properties from it along a particular path. Another important issue is gauge fixing. Suppose we are interested in studying the symmetry breaking of G into H . After symmetry breaking, strings with fluxes h and ghg^{-1} ($g \in G$) are typically *not* gauge equivalent to each other. To test whether symmetry breaking has occurred, one has to choose a field ϕ as a candidate for the Higgs field, gauge fix $\phi = \phi_0$ at x_0 and consider $H(\phi_0)$ and conjugacy classes and representations of H .

In the lattice formulation, it is convenient to put a string world sheet on a closed surface Σ^* on the dual lattice. Let Σ be the set of plaquettes threaded by Σ^* . Now, for each plaquette P in Σ , we choose a path, l_P , that runs from the base point x_0 to a corner of the plaquette [4]. Calibration of the plaquette is done along the path $l_P P l_P^{-1}$. More concretely, suppose that the plaquette action is

$$S_{gauge, P}^{(R)} = -\beta \chi^{(R)}(U_P) + c.c. \quad (5)$$

where R is some representation of the gauge group that defines the theory. The insertion of $F_a(\Sigma^*, x_0, \{l_P\}, \phi_0)$ modifies the action on each plaquette in Σ to

$$S_{gauge,P}^{(R)} \rightarrow -\beta \chi^{(R)} (V_{l_P} a V_{l_P}^{-1} U_P) + c.c. \quad (6)$$

where

$$V_{l_P} = \prod_{l \in l_P} U_l. \quad (7)$$

Now we turn to the operator which introduces classical charges into the system. Having gauge fixed the Higgs candidate $\phi = \phi_0$ at x_0 , all information of non-Abelian Aharonov-Bohm effect is encoded in the *untraced* Wilson loop operator

$$U^{(\nu)}(C, x_0, \phi_0) = D^{(\nu)} \left(\prod_{l \in C} U_l \right) \quad (8)$$

where C is a closed loop based at x_0 , ν is an irreducible representation of the gauge group G . The matrix elements of $U^{(\nu)}(C, x_0, \phi_0)$ can, in principle, be determined by interfering charged particles in the representation ν that traverse C with those that stay at the base point [12,13]. Just like F_a , the operator $U^{(\nu)}(C, x_0, \phi_0)$ is not gauge invariant.

If we did not gauge fix $\phi = \phi_0$ at the base point, global gauge transformations by the group G would be allowed. Thus, by the Schur's lemma, $\langle U^{(\nu)}(C, x_0) \rangle = \lambda I$. Notice that an irreducible representation of G is typically reducible in H . The result $\langle U^{(\nu)}(C, x_0) \rangle = \lambda I$ means that it would not be possible to resolve the various irreducible representations of H in the decomposition of an irreducible representation of G . This is clearly wrong. We conclude that it is crucial to perform gauge fixing.

Returning to the operator F , so far we have been vague about the choice of $\{l_P\}$. As it turns out, this is of greatest importance. It was noted in Ref. [4] that in a phase with free G charges, and in the leading order of weak coupling perturbation theory, the operator

$$\begin{aligned} \langle A_a^\nu(\Sigma^*, x_0, \{l_P\}; C) \rangle &= \frac{\langle F_a(\Sigma^*, x_0, \{l_P\}) U^{(\nu)}(C, x_0) \rangle}{\langle F_a(\Sigma^*, x_0, \{l_P\}) \rangle \langle tr U^{(\nu)}(C, x_0) \rangle} \\ &= \frac{1}{n_\nu} D^\nu \left(a^{k(\Sigma^*, C)} \right) \end{aligned} \quad (9)$$

where $k(\Sigma^*, C)$ is the linking number of the surface Σ^* and the loop C and the limit that Σ^* and C are infinitely large and far away is taken. (Note that Alford *et al.* overlooked the

importance of gauge fixing in their definitions of F_a and U^ν .) However, owing to quantum fluctuations, higher order terms in the weak coupling expansion may spoil this result [4]. The dominant contribution in a weak coupling expansion comes from configurations with a low density of frustrated plaquettes (i.e., a low density of virtual string loops). Alford *et al.* implicitly chose the long tails, $\{l_P\}$, from the plaquettes of Σ in such a way that all of them finally merge together at some point y_0 which is far away from the base point and is not on the Wilson loop. Unfortunately, this choice is vulnerable to quantum fluctuations. Consider in (2+1) dimensions a virtual vortex-antivortex pair whose worldline is non-trivially linked to the union of three objects: the Wilson loop, the tails and the string loop under calibration. (FIG. 1.) This will conjugate the measured flux relative the calibrated value. In the weak coupling expansion, such a configuration has a single excited link on the path that connects x_0 to Σ^* . This causes (in three spacetime dimensions) the excitation of four plaquettes and is suppressed by terms that are *independent* of the size of Σ^* and C or the separation between them. Thus, higher order corrections render the flux uncertain up to conjugation and this operator is useless as an order parameter. This was the conclusion drawn by Alford *et al.* [4].

Such a conclusion is unwarranted as it is based on their implicit choice of $\{l_P\}$. Any useful Aharonov-Bohm experiment to determine the flux of a string loop necessarily proceeds in two stages: calibration (with the operator $F(\Sigma^*)$) and subsequent measurement (with the operator $W(C)$). Both stages involve interference experiments with two beams of charged particles one of which traverses the string loop while the other just sits at the base point. To construct an order parameter for non-Abelian gauge theories, the effects of virtual string loops need to be considered. In the choice made by Alford *et al.*, the particles used for measurement have their worldlines along the Wilson loop C whereas the particles for calibration are kept in another box whose worldline runs along the long chain from x_0 and y_0 that are common to all the tails, but distinct from the Wilson loop. In other words, the particles for calibration and those for measurement are stored in separate boxes. It is only because of the decoupling of the two processes that quantum fluctuations can spoil the

results. The key issue is that repeated Aharonov-Bohm experiments do not necessarily agree. A virtual vortex-antivortex pair may spontaneously nucleate, wrap around the box which contains the calibrating particles and annihilate, thus changing the state of the calibrating particles relative to the ones used for subsequent measurement. Since the calibrating and measuring particles are in different states, the outcomes of the Aharonov-Bohm experiments done with these two different types of particles will generally be different.

To be more precise, a particle initially in a pure state $|u\rangle$ becomes a mixed state

$$\rho = \frac{1}{n_{[b']}} \sum_{b \in [b']} D^\nu(b) |u\rangle \langle u| D^\nu(b^{-1}), \quad (10)$$

after traversing a charge-zero string loop in the conjugacy class $[b']$. If this particle then travels around an a -vortex, the result of the interference experiment appears to show that the flux of the vortex is an incoherent superposition of $b^{-1}ab$ for $b \in [b']$ [4]. In conclusion, test particles may fail to recover the calibrated flux a of the classical string that the operator $F_a(\Sigma^*)$ introduces.

The resolution is simple. Keep the particles for both calibration and subsequent measurement in the *same* box. In the definition of F , calibration is done for all plaquettes threaded by the string loop. Physically, this means that we continuously calibrate the string loop by sending particles around it. For the subsequent measurement, one beam of particles should be kept in the box while the other beam winds around the calibrated string. (The two beams are wave packets.) The lattice realization of our choice of $\{l_P\}$ is shown in FIG. 2. Now a large portion of the Wilson loop close to x_0 represents the worldline of the box containing all test particles. Since continuous calibration of the flux of string is done by sending particles around the string loop, the tails (calibration paths) are chosen in such a way that many of them beginning from the base point are initially on the Wilson loop and branch out one by one from it. In Ref. [5], we argue rigorously that this construction overcomes all difficulties caused by quantum fluctuations.

It is crucial not to send out the two beams of particles for subsequent measurement too closely spaced in time. Otherwise, virtual string loops may wrap around the two beams,

thus changing the particles for measurement without affecting those for calibration. Virtual string loops can, of course, wind around the box containing all the particles, but this will affect both the calibration and measurement processes and lead to no net change. It is also possible for virtual string loops to wind around one of the two beams, say, that for the measurement. However, this incoherent effect will go away on average if we are willing to repeat independent identical experiments many times.

As remarked before, it is of utmost importance to gauge fix $\phi = \phi_0$ at the base point in the definitions of W and F . With our new choice of $\{l_P\}$ and gauge fixing, detailed arguments in Ref. [5] verify the non-analytical behavior of the operator

$$\langle A_a^\mu(\Sigma^*, x_0, \phi_0, \{l_P\}; C) \rangle = \frac{\left\langle \frac{1}{|H|} \sum_{h \in H(\phi_0)} F_{hah^{-1}}(\Sigma^*, x_0, \{l_P\}, \phi_0) \text{tr} U^\mu(C, x_0, \phi_0) \right\rangle}{\left\langle \frac{1}{|H|} \sum_{h \in H(\phi_0)} F_{hah^{-1}}(\Sigma^*, x_0, \{l_P\}, \phi_0) \right\rangle \langle \text{tr} U^\mu(C, x_0, \phi_0) \rangle}. \quad (11)$$

Note that we sum over only the elements and an irreducible representation μ of H . By dealing with the subtleties due to quantum fluctuations and gauge fixing squarely, we see clearly [5] how a gauge group G is reduced to an effective symmetry group H at low energies. Subsequent symmetry breaking of H can be studied in a similar manner.

In conclusion, we have constructed a set of order parameters for non-Abelian gauge theories. The study of the symmetry breakdown of S_3 into Z_2 is sketched in Ref. [5], which also discusses the *coherent* insertion of two or more string loops and contains derivations of many results stated in this Letter.

We are indebted to J. Preskill for bringing the issue of quantum fluctuations to our attention and for stimulating discussions. Numerous helpful conversations with S. Adler, M. Bucher, H. F. Chau, R. Dick, K.-M. Lee, J. March-Russell, P. McGraw, F. Wilczek, M. de Wild Propitius and P. Yi are also gratefully acknowledged. This work was supported in part by DOE DE-FG02-90ER40542.

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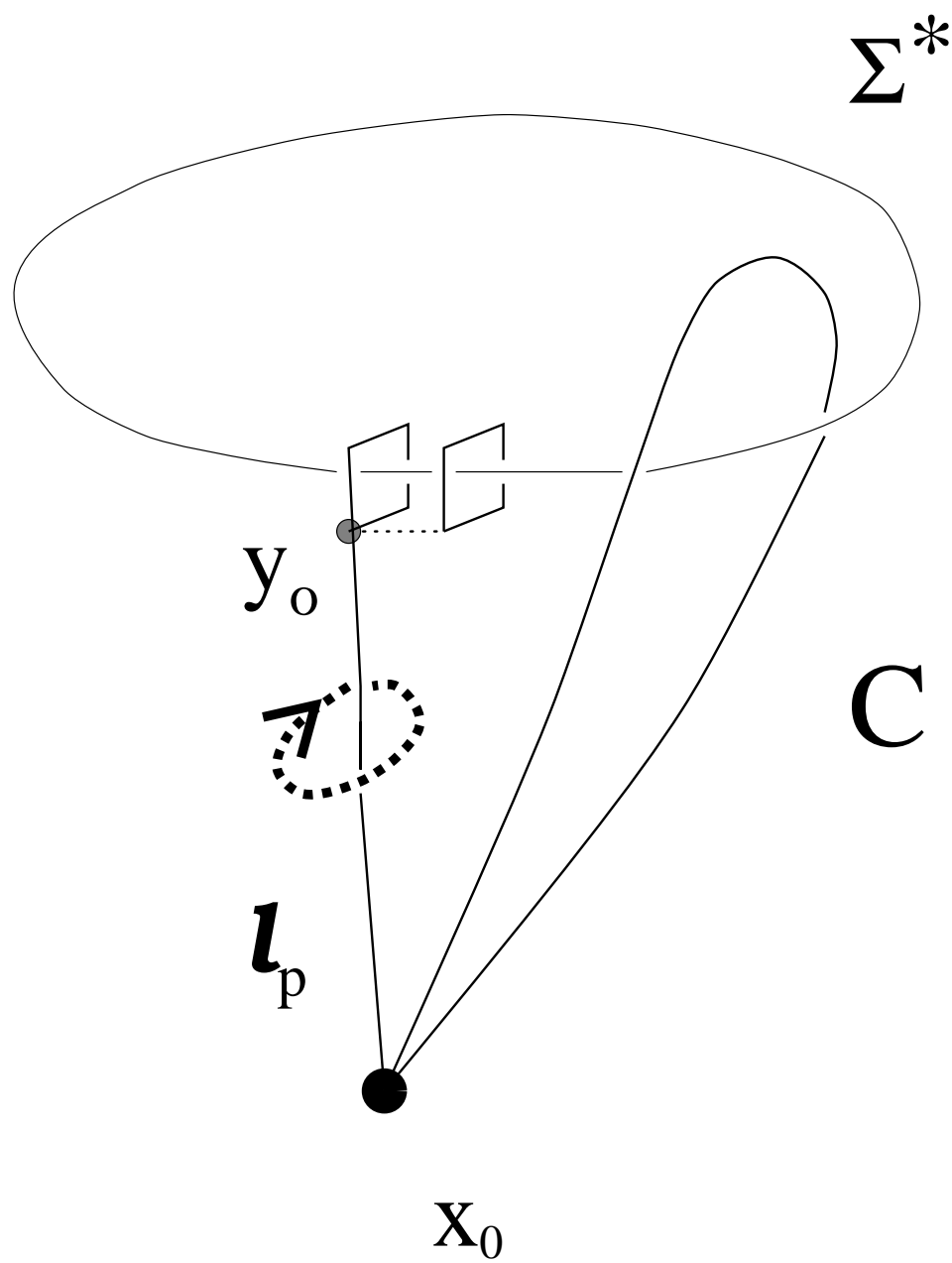
FIGURES

FIG. 1. Suppose all l_P merge together at some point y_0 *not* on the Wilson loop before reaching the base point. A worldline of virtual vortex conjugates the calibrated flux.

FIG. 2. Lots of long tails initially lie on the Wilson loop C . They eventually branch out from it one by one and never intersect one another afterwards. Moreover, the Wilson loop never comes close to retracing itself. To conjugate the calibrated flux without affecting the measurement, a virtual string loop must wrap around each tail after the branching out of each tail from the Wilson loop. Such a configuration becomes energetically costly as C and Σ^* become large.

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Σ^*

